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Weak convergence of finite graphs, integrated density of states and a Cheeger type inequality

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Abstract

In [G. Elek, On limits of finite graphs, Combinatorica, in press, URL: http://www.arxiv.org/pdf/math.CO/0505335] we proved that the limit of a weakly convergent sequence of finite graphs can be viewed as a graphing or a continuous field of infinite graphs. Thus one can associate a type II_1-von Neumann algebra to such graph sequences. We show that in this case the integrated density of states exists, that is, the weak limit of the spectra of the graph Laplacians of the finite graphs is the KNS-spectral measure of the graph Laplacian of the limit graphing. Using this limit technique we prove a Cheeger type inequality for finite graphs.

Keywords: Weak convergence of graphs; von Neumann algebras; Isoperimetric inequalities; Integrated density of states

1. Introduction

1.1. Weak convergence and limits of colored graph sequences

First let us recall some of the definitions and the main result from [4]. A *rooted colored d-graph* is a finite simple graph G with the following properties:

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- G has a distinguished vertex (the root).
- For any $p \in V(G)$, $\deg(p) \leqslant d$.
- The edges of G are properly colored by the colors $c_1, c_2, \ldots, c_{d+1}$. That is for each vertex $q \in V(G)$ the outgoing edges from q are colored differently. Note that by Vizing's theorem each graph with vertex degree bound d has such an edge-coloring.

A *rooted* (r,d)*-ball* is a rooted colored d-graph such that $\sup_{y \in V(G)} d_G(x,y) \leqslant r$, where x is the root of G and d_G denotes the shortest path distance. Two rooted colored d-graphs G and H are rooted isomorphic if there exists a graph isomorphism between them preserving the colors and mapping one root to the other one. We denote by $\mathcal{C}^{r,d}$ the finite set of rooted isomorphism classes of rooted (r,d)-balls. Now if G is a rooted colored d-graph and $\mathcal{A} \in \mathcal{C}^{r,d}$, then $T(G,\mathcal{A})$ denotes the set of vertices v in $V(G)$ such that $\mathcal{A}$ represents the rooted isomorphism class of the r-neighborhood of v, $B_r(v)$. Set $p_G(\mathcal{A}) := \frac{T(G,\mathcal{A})}{|V(G)|}$. That is G determines a probability distribution on $\mathcal{C}^{r,d}$, for any $r \geqslant 1$. Now let $\{G_n\}_{n=1}^{\infty}$ be a sequence of finite simple connected graphs with vertex degrees bounded by d, edge-colored by the colors $c_1, c_2, \ldots, c_{d+1}$. Suppose that $V(G_n) \to \infty$. Following Benjamini and Schramm [2] we say that $\{G_n\}_{n=1}^{\infty}$ is *weakly convergent* if for any $r \geqslant 1$ and $\mathcal{A} \in \mathcal{C}^{r,d}$, $\lim_{n\to\infty} p_{G_n}(\mathcal{A})$ exists.

Now let X be a compact metric space with a probability measure μ. Suppose that $T_1, T_2, \ldots, T_{d+1}$ are measure preserving continuous involutions on X satisfying the following conditions:

- For any $p \in X$, $T_i(p) = p$, for at least one i.
- If $T_i(p) = T_j(p)$, $i \neq j$, then $T_i(p) = T_j(p) = p$.

We call such a system $\mathcal{G} = \{X, T_1, T_2, \ldots, T_{d+1}, \mu\}$ a *d-graphing*. A d-graphing determines an equivalence relation on the points of X. Simply, $x \sim_{\mathcal{G}} y$ if there exists a sequence of points $\{x_1, x_2, \ldots, x_m\} \subset X$ such that

- $x_1 = x$, $x_m = y$,
- $x_{i+1} = T_j(x_i)$ for some $1 \leqslant j \leqslant d+1$.

Thus there exist natural simple (generally infinite) graph structures on the equivalence classes, the *leafgraphs*. Here x is adjacent to y, if $x \neq y$ and $T_j(x) = y$. By our conditions, all the leafgraphs have vertex degrees bounded by d and are naturally edge-colored (the (x,y) edge is colored by c_j if $T_j(x) = y$). Now if $\mathcal{A} \in \mathcal{C}^{r,d}$, we denote by $p_{\mathcal{G}}(\mathcal{A})$ the μ-measure of the points x in X such that the rooted r-neighborhood of x in $\mathcal{A}$ in its leafgraph is isomorphic to $\mathcal{A}$ as rooted (r,d)-ball. We say that $\mathcal{G}$ is the limit graphing of the weakly convergent sequence $\{G_n\}_{n=1}^{\infty}$, if for any $r \geqslant 1$ and $\mathcal{A} \in \mathcal{C}^{r,d}$

$$\lim_{n\to\infty} p_{G_n}(\mathcal{A}) = p_{\mathcal{G}}(\mathcal{A}).$$

In [4] we proved the following result.

Theorem 1. *If $\{G_n\}_{n=1}^{\infty}$ is a weakly convergent system of finite connected d-graphs as above, then there exists a d-graphing $\mathcal{G}$ such that for any $r \geqslant 1$ and $\mathcal{A} \in \mathcal{C}^{r,d}$, $\lim_{n\to\infty} p_{G_n}(\mathcal{A}) = p_{\mathcal{G}}(\mathcal{A})$.*

1.2. Cheeger type isoperimetric inequalities

Now let us recall two basic Cheeger-type isoperimetric inequalities for graphs. Let $G(V,E)$ be a finite connected graph. For a finite connected spanned subgraph $A \subseteq G$ we denote by ∂A the set of vertices x in $V(A)$ such that $x \in V(A)$, but there exists $y \notin V(A)$ that x and y are adjacent vertices. The Cheeger constant $h(G)$ is defined as

$$h(G) := \left\{ \inf \frac{|\partial A|}{|A|} \;\Big|\; A \subset G,\ \left|V(A)\right| \leqslant \frac{|V(G)|}{2} \right\}.$$

The classical isoperimetric inequality can be formulated in the following way: For any $\epsilon > 0$ and $d \in \mathbb{N}$, there exists a real constant $C(\epsilon, d) > 0$ such that if G is a finite connected graph with vertex degree bound d and $\lambda_1(G) \leqslant C(\epsilon, d)$ then $h(G) \leqslant \epsilon$. Here $\lambda_1(G)$ is the first non-zero Laplacian eigenvalue of G.

Now let $G(V,E)$ be an infinite connected graph with bounded vertex degrees. Then the isoperimetric constant of G is defined as follows:

$$i(G) := \left\{ \inf \frac{|\partial A|}{|A|} \;\Big|\; A \subset G,\ A \text{ is a finite connected spanned subgraph} \right\}.$$

Then again we have a Cheeger-type inequality (see e.g. [11, Theorem 4.27]): For any $\epsilon > 0$ and $d \in \mathbb{N}$, there exists a real constant $D(\epsilon, d) > 0$ such that if G is an infinite connected graph with vertex degree bound d and

$$\left[0, D(\epsilon, d)\right] \cap \operatorname{Spec} \Delta_G \neq 0,$$

then $i(G) \leqslant \epsilon$. That is, small values in the spectrum indicate the existence of some subsets with small boundaries. The main result of our paper can be stated informally in the following way: The abundance of small eigenvalues in a finite graph indicates the existence of a large system of disjoint subsets with small boundaries. To formulate our result precisely we need some definitions. For $\delta > 0$ and a finite connected graph G let $S(G, \delta)$ denote the set of Laplacian eigenvalues (with multiplicities) not greater than δ. Then

$$s(G, \delta) := \frac{|S(G, \delta)|}{|V(G)|}.$$

For a finite, connected graph G, $\epsilon > 0$ and $k \in \mathbb{N}$ we denote by $H(G, \epsilon, k)$ the set of vertices in G which can be covered by a connected spanned subgraph A such that $|A| \leqslant k$ and $\frac{|\partial A|}{|A|} \leqslant \epsilon$. Then $h(G, \epsilon, k) := \frac{|H(G,\epsilon,k)|}{|V(G)|}$. Also, we denote by $M(G, \epsilon, k)$ the cardinality of the maximal disjoint system of connected spanned subgraphs A in $V(G)$ such that $|A| \leqslant k$ and $\frac{|\partial A|}{|A|} \leqslant \epsilon$. We set $m(G, \epsilon, k) := \frac{M(G,\epsilon,k)}{|V(G)|}$. Clearly, there exists a constant $c(\epsilon, d, k) > 0$ such that for all graphs G with vertex degree bound d,

$$m(G, \epsilon, k) \geqslant c(\epsilon, d, k) h(G, \epsilon, k).$$

Using the weak limit technique we prove the following result.

Theorem 2. *For any $\epsilon > 0$ and $d \in \mathbb{N}$, there exists a constant $E(\epsilon, d) > 0$, such that the following holds: For any $t > 0$, there exist $k(t), n(t) \in \mathbb{N}$ and $s(t) > 0$ such that if $|V(G)| \geqslant n(t)$ holds for a finite connected graph G with vertex degree bound d, and $s(G, E(\epsilon, d)) \geqslant t$, then $m(G, \epsilon, k(t)) \geqslant s(t)$.*

The main idea is to associate a type II_1-von Neumann algebra for a weakly convergent graph sequence and prove that the integrated density of states exists, that is, the spectra of the graph Laplacians of the finite graphs converge weakly to the so-called KNS-spectrum of the graph Laplacian on the limit graphing.

Note that in [8] Lovász and Szegedy introduced the notion of weak convergence and the limit object for dense graph sequences. They used the limit technique to give a new proof of a theorem of Alon and Shapira in [9]. Another application of the limit technique can be found in the paper of Aldous and Steele [1].

2. The KNS-measure

First let us recall the classical definitions of measurable fields of Hilbert-spaces and operators [3]. Let Z be the standard Borel space with a probability measure μ. Let $\mathcal{H} = l^2(\mathbb{N})$ be the complex separable infinite dimensional Hilbert-space with orthonormal basis $\{e_i\}_{i=1}^{\infty}$. A measurable field of Hilbert-spaces is given by a sequence $\{f_n\}_{n=1}^{\infty} \subset L_2(Z,\mu)$ such that $\sum_{n=1}^{\infty} \int_Z |f_n(x)|^2\, d\mu(x) < \infty$. By the Beppo–Levi Theorem for almost every $x \in Z$: $f_x = \sum_{n=1}^{\infty} f_n(x)e_n \in \mathcal{H}$. We denote the space of the measurable fields of Hilbert spaces by $\int_Z \mathcal{H}_x\, d\mu(x)$, which is a Hilbert-space with respect to the pointwise inner product.

A field of bounded linear operators is a $\mathcal{B}(\mathcal{H})$-valued function on Z such that $x \to \langle A_x(e_i), e_j\rangle$ is a measurable function for all i, j and $\operatorname{Ess\,sup}_{x\in Z} \|A_x\| < \infty$. Such fields of operators form the von Neumann algebra $\int_Z \mathcal{B}(\mathcal{H})\, d\mu(x)$, with pointwise addition, multiplication and $*$-operation. Note that $\int_Z A_x\, d\mu(x) \in \int_Z \mathcal{B}(\mathcal{H})\, d\mu(x)$ is invertible if and only if A_x^{-1} exists for almost every $x \in Z$ and $\operatorname{Ess\,sup}_{x\in Z} \|A_x^{-1}\| < \infty$.

Now we introduce the notion of continuous field of graphs associated to a d-graphing. Let $\mathcal{G} = (X, T_1, T_2, \dots, T_{d+1}, \mu)$ be a d-graphing of our Theorem 1. For each $r \geqslant 1$, and $\mathcal{A}_r \in \mathcal{C}^{r,d}$ we label the vertices of $\mathcal{A}_r$ by natural numbers inductively, satisfying the following conditions:

- The root is labeled by 1.
- Any vertex has different labeling.
- The labeling of $\mathcal{A}_r$ is compatible with the labeling of $\mathcal{A}_{r-1}$.
- If $d_G(x,y) < d_G(x,z)$, then the label of y is smaller than the label of z.
- If $|\mathcal{A}_r| = k$, then the labels of the vertices is exactly $\{1, 2, \dots, k\}$.

Therefore for each $x \in X$ we associate an infinite graph with edge-coloring by $c_1, c_2, \dots, c_{d+1}$ and an extra vertex labeling by the natural numbers. Of course this graph is isomorphic to the leafgraph of x and each vertex can also be viewed as an element of X.

This is the *continuous field of infinite graphs* associated to the graphing. The total vertex set of this field is $\mathcal{R} \subset X \times X$, $(x,y) \in \mathcal{R}$ if $x \sim y$, the equivalence relation given by the graphing. The space $\mathcal{R}$ is equipped with the counting measure ν [5] hence using the vertex labels one can view $L_2(\mathcal{R},\nu)$ as $\int_X l^2(V(\mathcal{G}_x))\, d\mu(x)$.

The leafwise Laplacian operator $\int_X \Delta_{\mathcal{G}_x}\, d\mu(x)$ is a measurable field of bounded operators on $\int_X l^2(V(G_x))\, d\mu(x)$. Feldman and Moore [5] introduced an important subalgebra of the algebra of the fields of bounded operators in the case of graphings. We briefly review their construction. Consider the space of bounded measurable functions $\kappa : \mathcal{R} \to \mathbb{C}$ with *finite bandwidth*, that is for some constant $b_\kappa > 0$ depending only on κ: $\kappa(x,y) = 0$ if $d_{\mathcal{G}}(x,y) > b_\kappa$. Here $d_{\mathcal{G}}(x,y)$

denotes the shortest path-distance on the leaf-graph of x. One can associate a measurable field of bounded operators to κ in the following way. If $f \in L^2(\mathcal{R}, \nu)$, then

$$T_\kappa(f)(x,y) = \sum_{z \sim x} f(x,z)\kappa(z,y).$$

These operators are called *random operators*. The weak closure of such operators T_κ in $\int_X \mathcal{B}(l^2(V(\mathcal{G}_x)))\,d\mu(x)$ is the Feldman–Moore algebra $\mathcal{N}_\mathcal{R}$. The point is that $\mathcal{N}_\mathcal{R}$ possesses a trace. It is a von Neumann algebra of type II_1.

The trace of T_κ is given by

$$\mathrm{Tr}_\mathcal{G}(T_\kappa) = \int_X \kappa(x,x)\,d\mu(x).$$

The leafwise Laplacian operator on $\mathcal{G}$, $\Delta_\mathcal{G}$ is clearly an element of $\mathcal{N}_\mathcal{R}$, given by a bounded measurable function of finite bandwidth, where

$$\Delta_\mathcal{G}(f)(x,y) = \deg(y)f(x,y) - \sum_{z,\ z \text{ is adjacent to } y} f(x,z).$$

We shall denote the Laplacians on the finite graphs G_i by Δ_{G_i}. (Of course the Laplacian does not depend on the edge coloring.)

Proposition 2.1. *For any $n \geqslant 1$:*

$$\lim_{i\to\infty} \frac{\mathrm{Tr}_i(\Delta^n_{G_i})}{|V(G_i)|} = \mathrm{Tr}_\mathcal{G}\bigl(\Delta^n_\mathcal{G}\bigr),$$

where $\mathrm{Tr}_i : \mathrm{Mat}_{V(G_i)\times V(G_i)}(\mathbb{C}) \to \mathbb{C}$ *is the usual trace.*

Proof. Obviously,

$$\mathrm{Tr}_i(\Delta_{G_i}) = \sum_{x\in V(G_i)} \deg(x)$$

and for the powers of the Laplacians:

$$\mathrm{Tr}_i\bigl(\Delta^n_{G_i}\bigr) = \sum_{x\in V(G_i)} s_n(x),$$

where $s_n(x)$ depends only on the isomorphism class of the n ball around x in G_i. Hence

$$\frac{\mathrm{Tr}_i(\Delta^n_{G_i})}{|V(G_i)|} = \sum_{A\in\mathcal{C}^{n,d}} p_{G_i}(\mathcal{A})s_n(\mathcal{A}),$$

where $s_n(\mathcal{A})$ is the value of s_n at the root. On the other hand,

$$\mathrm{Tr}_\mathcal{G}\bigl(\Delta^n_\mathcal{G}\bigr) = \sum_{\mathcal{A}\in\mathcal{C}^{n,d}} p_\mathcal{G}(\mathcal{A})s_n(\mathcal{A}).$$

Hence our proposition follows. □

By our vertex bound condition $\mathrm{Spec}(\Delta_\mathcal{R})$ and $\mathrm{Spec}(\Delta_{G_i})$ for all i are contained in some interval $[0,l]$. Recall that the spectral measure λ_T of a positive self-adjoint operator A on the finite dimensional Euclidean space $\mathbb{C}^n$ is a point measure on $[0,\infty)$ defined as follows:

$$\lambda(A) = \frac{\#(\text{the eigenvalues of } T \text{ with multiplicities in } A)}{n}.$$

By the classical (finite dimensional) spectral theorem

$$\int_0^l x^n \, d\lambda_{\Delta_{G_i}}(x) = \frac{\mathrm{Tr}_i(\Delta^n_{G_i})}{|V(G_i)|}.$$

On the other hand, by the von Neumann's spectral theorem

$$\int_0^l x^n \, d\lambda_{\Delta_{\mathcal{R}}}(x) = \mathrm{Tr}_{\mathcal{G}}\bigl(\Delta^n_{\mathcal{G}}\bigr),$$

where $\lambda_{\Delta_{\mathcal{G}}}[0,x] = \mathrm{Tr}_{\mathcal{G}}(E_{\Delta_{\mathcal{G}}})([0,x])$ for the projection-valued measure $E_{\Delta_{\mathcal{G}}}$ associated to the bounded self-adjoint operator $\Delta_{\mathcal{G}}$. The measure $\lambda_{\Delta_{\mathcal{G}}}$ is called the KNS-measure associated to the Laplacian $\Delta_{\mathcal{G}}$ (see [6] for a discussion on spectral measures).

That is for any polynomial $p \in \mathbb{C}[x]$,

$$\lim_{i\to\infty} \int_0^l p(x)\, d\lambda_{\Delta_{G_i}}(x) = \int_0^l p(x)\, d\lambda_{\Delta_{\mathcal{R}}}(x).$$

Therefore we proved that the integrated density of states exists (see [7] for a discussion on integrated density of states and random operators).

Theorem 3. *If the sequence of graphs $\{G_i\}_{i=1}^\infty$ weakly converges to the graphing $\mathcal{G}$, then the associated spectral measures $\lambda_{\Delta_{G_i}}$ weakly converge to $\lambda_{\Delta_{\mathcal{R}}}$ (see also* [10]).

3. The proof of Theorem 2

Let us choose $E(\epsilon,d) := \frac{1}{2}D(\epsilon,d)$, where $D(\epsilon,d)$ is the constant in the isoperimetric inequality for infinite graphs. Suppose that the theorem does not hold. Then there exists $t>0$ with the following property: For any k, one can choose a sequence $\{G_n^k\}_{n=1}^\infty$ of finite graphs with vertex degrees bounded by d such that

- $|V(G_n^k)| \to \infty$,
- $s(G_n^k, \frac{1}{2}D(\epsilon,d)) \geqslant t$,
- $\lim_{n\to\infty} m(G_n^k,\epsilon,k) = 0$.

Since if $l>k$ then $m(G,\epsilon,k) \leqslant m(G,\epsilon,l)$, we can choose a weakly convergent sequence of finite connected colored d-graphs $\{H_n\}_{n\geqslant 1}^\infty$ such that:

- $|V(H_n)| \to \infty$,
- $s(H_n, \frac{1}{2}D(\epsilon,d)) \geqslant t$,
- $\lim_{n\to\infty} m(H_n,\epsilon,k) = 0$ for any k.

By the argument in the Introduction one can see that $\lim_{n\to\infty} h(H_n,\epsilon,k) = 0$ as well.

Therefore if L is a fixed finite connected graph and $\mathcal{A}_r$ $(\mathrm{diam}(L) < r)$ contains a spanned subgraph L' isomorphic to L, $\frac{|\partial L'|}{|L'|} \leqslant \epsilon$ such that the root is in L', then $\lim_{n\to\infty} p_{H_n}(\mathcal{A}_r) = 0$. Consequently, if one considers the limit graphing $\mathcal{G}$ of Theorem 1, then for each leafgraph $\mathcal{G}_x$,

$i(\mathcal{G}_x) \geqslant \epsilon$. That is for each leafgraph the spectral gap of the Laplacian at the zero is greater than $\frac{3}{4}D(\epsilon, d)$. Therefore, the measurable field of operators $(\int_X \Delta_{\mathcal{G}_x}\, d\mu(x) - \lambda)$ is invertible, with a uniformly bounded inverse, whenever $\lambda \leqslant \frac{1}{2}D(\epsilon, d)$. In other words

$$\operatorname{Spec}\left(\int_X \Delta_{\mathcal{G}_x}\, d\mu(x)\right) \cap \left[0, \frac{1}{2}D(\epsilon, d)\right] = \emptyset.$$

Now we use our Theorem 3. The spectrum of $\int_X \Delta_{\mathcal{G}_x}\, d\mu(x)$ is the same in the von Neumann algebra $\int_X \mathcal{B}(l^2(\mathcal{G}_x))\, d\mu(x)$ as in the Feldman–Moore subalgebra. However by Theorem 3, the KNS-measure of the interval $[0, \frac{1}{2}D(\epsilon, d)]$ is at least t. This leads to a contradiction.

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